FORMULATION AND APPLICATION OF MIXED ELEMENTS FOR THE 2D-BOUNDARY ELEMENT METHOD (BEM)

Klaus Schimmanz[+] , Arnulf Kost[++]

[+] Chair of Numerical and Applied Mathematics, TU Cottbus, Universitätsplatz 3-4, 03044 Cottbus, Germany
[++] Chair of Fundamentals in Electrical Engineering and Computational Electromagnetics, TU Cottbus, Universitätsplatz 3-4, 03044 Cottbus, Germany

Abstract

Accuracy and time consumption in numerical computations are often in contradiction to each other. In modern industrial design processes flux field computation become more and more important. Thus, it is desirable to minimize any methodical error for better performance. This article discusses a flaw of frequently used standard elements in BEM and shows a way to avoid it.

Introduction

The boundary integral equation required by the Boundary Element Method (BEM) can be deduced in a simple way based on considerations of weighted residuals. In the case of the Laplacian differential equation, for instance, it leads to the following expression (see [1]):

$$u_i = \oint_\Gamma \left(\omega_i \frac{\partial u}{\partial n} - u \frac{\partial \omega_i}{\partial n}\right) d\Gamma \qquad (1)$$

where Γ is domains contur, u is boundaries potential function on Γ, and n the outward vector unit normal. ω and $\partial\omega/\partial n$ are the so called fundamental solutions of the partial differential equation. The index i indicates an arbitrary location within the domain or on the contur. To determine a potential somewhere in the domain by (1) it is necessary to know the whole potential function u along Γ and its directional derivative $\partial u/\partial n = \boldsymbol{n}\nabla u$. Some of them are given in sections by the necessary boundary conditions of the problem to be solved. But the missing ones need to be computed first. As well known, this can be accomplished by dividing the boundary Γ into short segments, the elements, which leads to a finite set of equations and thus, to a linear algebraic equation system. For each element interpolation formulations of the potential functions and directional derivatives with some free parameters are used and the unknown values of the equation system will be these parameters.

Assume for simlicity that the body is two dimensional and polygonal bounded. The points where the unknown values are considered are called nodes and taken to be in the middle of the element for the so called constant elements (Fig. 1, left).

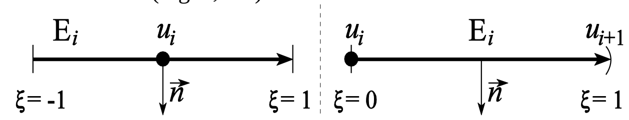

Fig. 1 Elements: *left:* constant standard; *right:* constant-linear mixed

The values of u and $\partial u/\partial n = \boldsymbol{n}\nabla u$ are assumed to be constant over each constant standard element and equal to the mid-element node value. For standard elements the same interpolation function type is employed for both value approximations. But exactly this is the reason for a 'built in flaw' of the BEM which causes distortions of computed potential fields. Gradient calculations close to a standard element boundary, for instance, lead in the case of constant elements to (see also [1]):

$$u_i = const. \;\Rightarrow\; \frac{\partial u_i}{\partial t}\boldsymbol{t} \equiv 0 \;\Rightarrow\; \nabla u_i = \frac{\partial u_i}{\partial n}\boldsymbol{n} + \frac{\partial u_i}{\partial t}\boldsymbol{t} = \frac{\partial u_i}{\partial n}\boldsymbol{n} = const.$$

where $\boldsymbol{t}$ is tangential vector unit normal of the element. Thus, there is an additional error and this one is directionally dependent. Similar effects like this are happening to all standard elements. To avoid this problem a different formulation for value interpolations is obligatory.

Mixed Element Formulation

The reason for disappearing tangential components in the case of constant standard elements is the higher polynomial approximation order of the normal component. The order difference is exact one. This will be the same for any other higher standard element type, too. Therefore, suppressing this disturbance means in the simplest case to apply a constant-linear mixed formulation like:

$$u(\xi) = (1 - \xi)u_i + \xi u_{i+1} \text{ , and } \frac{\partial u(\xi)}{\partial n} = u_i' = const. \qquad (2)$$

where $\xi \in [0; 1)$ is the local coordinate of the element. The node has to be located at one element bound for this mixed element type (for (2) see Fig. 1, right). However, the following linear equation system can be over-determined by some sorts of boundary condition transitions which will need a special but simple treatment of the equation set.

Exemlary Comparison

Fig. 2 shows the numerical solution of the same homogenous potential field problem by using constant standard elements and by constant-linear mixed elements. In both cases just 4 elements are employed. More examples will be presented in the full paper version.

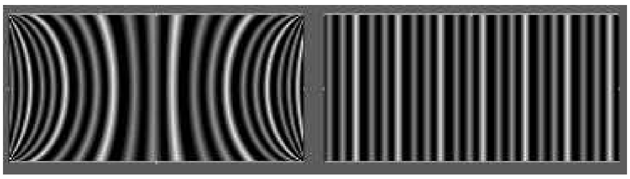

Fig.2 Computed potential field. *left:* constant standard elements; *right:* constant-linear mixed elements

Conclusions

Mixed elements shown as here provide a directionally independent approximation quality of the gradient and thus, a more homogeneous error distribution within the domain. For adaptive mesh refinement methods based on error estimations the property of error locality is an indispensable precondition. Furthermore, such mixed elements can reduce the number of necessary elements for field computations and thus, speed up the method. The integration effort of constant-linear elements shown here is $3(N^2\text{-}2N)$. This is more than for constant standard elements $2(N^2\text{-}N)$, but less as for linear standard elements $4(N^2\text{-}2N)$.

As well as for constant standard elements the integrals of constant-linear mixed elements can be carried out analytically (see [2]). This further reduces field confusions at border adjacencies as it is happening by often applied numerical Gaussian quadratur formulas.

References

[1] A. Kost, Numerische Methoden in der Berechnung elektromagnetischer Felder, Springer Verlag, Berlin, 1994.

[2] K. Schimmanz, Konzipieren und Bewerten von Hochpräzisions-Hybridwiderständen durch Laser-Trimm-Simulation, Dr.-Ing. thesis, TU Berlin, 2002.